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GROUP RINGS OF POLYCYCLIC GROUPS *

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1. Introduction

1.1. *Group rings with Max-r*

We say that a ring has *Max-r* if and only if it has the maximal condition on right ideals. The study of group rings with Max-r was begun by Philip Hall in 1954. A group whose integral group ring satisfies Max-r must necessarily satisfy *Max-s*, the maximal condition on subgroups. The only groups we know to have Max-s are the polycyclic by finite ones.

We recall that a group G is *polycyclic* if and only if there is a series

$$(1) \qquad 1 = G_0 \lhd G_1 \lhd \ldots \lhd G_{n-1} \lhd G_n = G$$

in which the factors G_{i+1}/G_i, $0 \leqslant i < n$, are all cyclic. Polycyclic by finite groups are those which have a normal polycyclic subgroup of finite index. What Hall proved in [4] was that the integral group rings of these known groups with Max-s must have Max-r. It is still an open question whether these are the only groups whose integral group rings have Max-r, and we shall say nothing more about this. Our methods depend upon structural information about polycyclic groups, which is not available for arbitrary groups with Max-s, and relate to the questions raised by Hall in his other two papers [5] and [6] about these groups.

1.2. *Simple modules*

We say that a field is *absolute* if and only if every one of its non-zero elements is a root of unity. Absolute, then, is short for absolutely algebraic of prime characteristic. As one of the main theorems of [5], Hall proved that if G is a finitely generated nilpotent group and if k is an absolute field, then simple kG-modules are finite-dimensional over k. He left open the question of whether this would still be true if G were polycyclic. We answer this with:

* Dedicated to Philip Hall, on the occasion of his seventieth birthday, 11 April 1974.

Theorem A. *If k is any absolute field and if G is any polycyclic by finite group then any simple kG-module has finite dimension over k.*

An equivalent way of putting this is to say that *an infinite polycyclic by finite group cannot have a faithful irreducible representation over an absolute field.*

As Hall pointed out in [5], if a polycyclic group G has no Abelian subgroups of finite index, then if K is any non-absolute field, there are simple KG-modules of infinite dimension over K. There is therefore not much room for improvement if we concern ourselves only with group algebras. However, the result can be applied to group rings over certain coefficient rings other than fields if we couple it with rather easier results. For example, together with [5, Lemma 2] it shows at once that *if G is a polycyclic by finite group, then every simple $\mathbf{Z}G$-module is finite.* More generally it applies to group rings over a *Hilbert ring*, that is, one which has Max-r and is such that the Jacobson radical of every homomorphic image is nilpotent.

Suppose that J is a commutative Hilbert ring and G a polycyclic by finite group. A by-product, Corollary C3, of the first part of the proof of Theorem A is that *if M is a simple JG-module, then M is killed by some maximal ideal L of J.* If we write K for the field J/L, then M is naturally a KG-module; and if it happens that K is absolute, then Theorem A shows that $\dim_K(M)$ is finite.

If we say that a simple homomorphic image of a ring is a *capital* of that ring, we may state:

Corollary A. *Suppose J is a commutative Hilbert ring all of whose capitals are absolute. If G is a polycyclic by finite group then any simple JG-module is finite-dimensional over a capital of J.*

Theorem A has been given two erroneous proofs already. That advanced by Zalesskiĭ in [17] was retracted*, and the first lemma of that given by Levič [11] is false.

1.3. *Monolithic modules*

Hall used his results to deduce theorems about finitely generated Abelian by nilpotent groups. It will be remembered that the main result of [5] concerned monolithic groups; a group being monolithic if it is non-trivial and if the intersection of all the non-trivial normal subgroups is non-trivial. We take this terminology over to modules: if M is an R-module, we say that M is *monolithic* if and only if M is non-zero and the intersection of all the non-zero submodules of M is non-zero. This intersection is called the *lith* of M. If M is monolithic, its lith is clearly a simple R-module. Hall's theorem that *monolithic finitely generated Abelian by nilpotent groups are finite* is easily seen to be equivalent with the fact that finitely generated monolithic modules for the integral group ring of a finitely generated nilpotent group are finite. He says explicitly in [5] that his results could easily be extended to finitely gener-

* See the review of [17] in Math. Rev. 44 (1972).

ated Abelian by polycyclic groups if it could be established that the simple modules for the integral group ring of a polycyclic group are all finite.

Let R be the integral group ring of a finitely generated nilpotent group G, and suppose that M is a finitely generated monolithic R-module with lith U. Let $\mathfrak{g}$ be the augmentation ideal of R. In proving that M is finite, an easy reduction allows one to assume that $U\mathfrak{g}$ is zero. Hall's arguments then show that some power of $\mathfrak{g}$ must kill M, and the result is an easy consequence of this fact.

Now it is certainly not true to say that if G is a polycyclic group and M is a finitely generated monolithic $\mathbf{Z}G$-module whose lith is killed by $\mathfrak{g}$, then M is killed by some power of $\mathfrak{g}$. Indeed, it will be clear from Section 5 that *the only polycyclic by finite groups for which this is true for all such monolithic modules are the nilpotent ones.* This suggests the possibility that Hall was in error, at least in saying that his results could be extended easily. Whether or not they can be extended at all we do not know.*

1.4. *The Nullstellensatz*

Suppose that (1) is a series for the polycyclic group G. If $G_{i+1} = G_i\langle x_{i+1}\rangle$ for $0 \leqslant i < n$, then the elements of G all have the form $x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}$ with integers $r_1, r_2, \dots, r_n$. They may be thought of as monomials in $x_1, x_2, \dots, x_n$ and their inverses; although the representation will not be unique in general, nor will the 'variables' necessarily commute with each other. However, group rings of G look sufficiently like polynomial rings for us to expect that the theorems which can be proved about them should resemble those which are true for polynomial rings. The Weak Nullstellensatz [1, p. 82] says that the simple modules of a polynomial ring $P = K[X_1, X_2, \dots, X_n]$ over a field K are all of finite dimension over K, so that Theorem A can be thought of as a Weak Nullstellensatz for $\not k G$. The analogy is not quite exact, because of the restriction on the field.

The Strong Nullstellensatz comes in various forms. The most convenient from our point of view is that discussed by Krull [10] and states that the polynomial ring P is a Hilbert ring. Our next results concern this and the questions asked by Hall in [6]. The main results of [6] were group-theoretic, but depended essentially on a version of the Nullstellensatz for integral group rings of finitely generated nilpotent groups. Let R be an image of such a ring. Hall proved that the elements of the centre of R which lay in the Jacobson radical of R had to be nilpotent. He doubted whether the same result would hold if R was replaced by an image of a group algebra over a general field, having in mind the failure of Theorem A for non-absolute fields. His doubt can now be dispelled. Not only does the thorem go over to group algebras, but it is true also for polycyclic groups. Suppose J is a commutative ring and G is a polycyclic by finite group. We shall prove, after Corollary C5, that *if J is a Hilbert ring, then so is JG.* This will be an easy consequence of the fact that a simple JG-module is killed by a maximal ideal of J. Of course, the property of being a Hilbert ring goes

* See the remark on p. . . .

to homomorphic images: since the coefficient ring is an image of JG, it is obvious that JG can be a Hilbert ring only if J is.

The more general version of the Nullstellensatz which Hall proved as [6, Lemma 9] had to do with finitely generated modules M for images R of an integral group ring of a finitely generated nilpotent group. He proved that if a central element of R killed every simple image of M, then some power of that element killed M. It is a simple consequence of the fact, proved in [15], that every ideal of R is polycentral that this result can be extended to show that *any ideal of R which kills every simple image of M has some power killing M.* Although Hall's [6, Lemma 9] remains true, by Corollary C5, if the group involved is polycyclic rather than nilpotent, the extension does not: *if G is any polycyclic by finite group which is not nilpotent, then there is a finite $\mathbf{Z}G$-module M and an ideal X of $\mathbf{Z}G$ which kills every simple image of M but has no power killing M.* Indeed X may be taken to be the ideal $\bar{\mathfrak{h}}$ of $\mathbf{Z}G$ generated by the augmentation ideal $\mathfrak{h}$ of any normal non-nilpotent subgroup H of G. On the other hand, we shall prove:

Theorem B. *Suppose that J is a commutative Hilbert ring all of whose capitals are absolute, and that H is a nilpotent normal subgroup of the polycyclic by finite group G. If $\mathfrak{h}$ kills every simple image of the finitely generated JG-module M, then some power of $\mathfrak{h}$ kills M.*

We leave open the question of whether $\mathfrak{h}$ can be replaced by an arbitrary ideal of JH.

By a *chief factor* of a module we mean a simple image of a submodule. Since, in Theorem B, the subgroup H is normal in G and $\mathfrak{h}^n$ kills M for some n, every chief factor of M is killed by $\mathfrak{h}$. If we begin with this as hypothesis, we may deduce that some $\mathfrak{h}^n$ kills M without the nilpotency of H. We state this as:

Corollary B. *Suppose that J is a commutative Hilbert ring all of whose capitals are absolute, and that H is a normal subgroup of the polycyclic by finite group G. If $\mathfrak{h}$ kills every chief factor of the finitely generated JG-module M, then some power of $\mathfrak{h}$ kills M.*

1.5. *Applications*

In describing these we shall write $\mathfrak{X}$ for the class of all finitely generated Abelian by polycyclic by finite groups. Since $\mathfrak{X}$ is an image-closed class, it follows from [5, Lemma 1] that every $\mathfrak{X}$-group is residually finite if and only if every monolithic $\mathfrak{X}$-group is finite. In turn, this is to say that finitely generated monolithic $\mathbf{Z}G$-modules are finite if G is polycyclic by finite. Since we cannot decide whether they are or not, the question of the residual finiteness of $\mathfrak{X}$-groups must remain. However, it is an immediate consequence of the fact that the simple modules are finite that *every chief factor of an $\mathfrak{X}$-group is finite.*

Our inability to decide the residual finiteness question does not prevent us from extending the results of [6] to $\mathfrak{X}$-groups. Let Γ be any $\mathfrak{X}$-group. There exists an Abelian normal subgroup M of Γ such that the factor group $G = \Gamma/M$ is polycyclic by finite. We may regard M as a $\mathbf{Z}G$-module via conjugation and apply Theorem B with $J = \mathbf{Z}$. Let Δ be a normal subgroup of Γ containing M, and suppose that H is the projection Δ/M of Δ in G. If H is nilpotent and if $\mathfrak{h}$ kills every simple image of M, then we may deduce that $M\mathfrak{h}^n = 0$ for some n. The multiplicative interpretation of this is that M is contained in the n^{th} term of the upper central series of Δ. Since H is nilpotent, it follows that Δ is nilpotent.

A first consequence of this is that *the intersection of the centralizers of the chief factors of an* $\mathfrak{X}$*-group is nilpotent.* For if Δ is this intersection then obviously $\mathfrak{h}$ kills every chief factor of M; and it follows from the work of Hirsch [8] that H is nilpotent.

A further consequence is that *the Frattini subgroup of an* $\mathfrak{X}$*-group is nilpotent.* Indeed, if Φ is the Frattini subgroup of the $\mathfrak{X}$-group Γ, and if Δ/Φ is the Fitting radical of Γ/Φ, then we may even say that Δ is nilpotent. For Hirsch [9] showed that Φ is nilpotent if Γ is polycyclic, and Gaschütz [3] that Δ is nilpotent if Γ is finite. It follows that the projection of Δ in any polycyclic by finite image of Γ is nilpotent. This shows that H is nilpotent and that $\mathfrak{h}$ kills the simple images of M. For if U is a maximal submodule of M, then M/U is finite by Theorem A. Therefore Γ/U is polycyclic by finite and Δ centralizes M/U.

Together with arguments from Hall [6], these results show that *[6, Theorem 2] holds for finitely generated groups which have a nilpotent by polycyclic subgroup of finite index.*

2. Theorem A – Reductions

2.1. *The general idea*

In [5], Hall proved that if G is any polycyclic by finite group and M any simple $\mathbf{Z}G$-module, then $Mp = 0$ for some rational prime p. Once this had been done, M became a $\mathfrak{k}G$-module with $\mathfrak{k} = \mathbf{Z}/p\mathbf{Z}$. With x an element of the centre of G, Hall then viewed M as a $\mathfrak{k}\langle x\rangle$-module and proved that M was killed by $x^n - 1$ for some positive n. That M was finite if G happened to be nilpotent by finite then followed without much further difficulty. He proved these two results by first examining the structure of finitely generated $\mathbf{Z}G$-modules as $\mathbf{Z}$-modules, and next by examining finitely generated $\mathfrak{k}G$-modules in terms of their structure as $\mathfrak{k}\langle x\rangle$-modules. The two steps were rather similar and depended, although not in any essential way, upon the fact that both $\mathbf{Z}$ and $\mathfrak{k}\langle x\rangle$ are principal ideal rings.

If G is an arbitrary polycyclic by finite group, it is not certain that there is any subgroup of finite index which has a non-trivial centre, and Hall's second step cannot be used. However, much of his argument can be generalized by replacing central elements x by Abelian normal subgroups A of G. Of course, any group ring $S = JA$ is embedded in a natural way in the corresponding group ring $R = JG$. The proofs of the main theorems depend on analyzing R-modules in terms of their structures as S-modules. Distinctions have to be made according to whether the modules are torsion-free for S or not. Induction techniques will deal with those which are not; and the main burden of the proof lies in the discussion of the torsion-free ones.

2.2. *R-modules which are torsion-free for S*

The first result, which is a slight generalization of [5, Lemmas 5.2 and 5.6], and which we shall prove in Section 3, is:

Theorem C. *Suppose J is a commutative Noetherian ring and R is the group ring of a polycyclic by finite group G over J. Suppose S is the group ring over J of an Abelian normal subgroup A of G. If M is a finitely generated R-module which is torsion-free as an S-module, then there exists a free S-submodule F of M and a non-zero ideal Λ of S such that every finitely generated S-submodule of M/F is killed by a product $\Lambda^{x_1}\Lambda^{x_2}\ldots\Lambda^{x_n}$ of conjugates of Λ under G.*

Since A is a normal subgroup of G, the mapping $\sigma \to x^{-1}\sigma x$ ($\sigma \in S$) is an automorphism of S for each x in G. We shall write σ^x for $x^{-1}\sigma x$ and give terms like conjugate, normalizer and centralizer the obvious meanings.

Suppose U is a finitely generated S-submodule of M. From Theorem C there is an ideal $X = \Lambda^{x_1}\Lambda^{x_2}\ldots\Lambda^{x_n}$ of S with $UX \leqslant F$. If L is a maximal ideal of S which contains none of the Λ^{x_i}, then it does not contain X so that $S = X + L$. Since $U = US$, it follows that $U \leqslant F + UL$. It also follows that $(UL) \cap F \leqslant FL$, for $((UL) \cap F)S \leqslant ULX + FL = UXL + FL$. Since ML is the set-theoretic union of all the UL, we may state:

Corollary C1. *If L is a maximal ideal of S which contains no conjugate of Λ, then $M = F + ML$ and $(ML) \cap F = FL$.*

We shall discuss most of the consequences of this in Section 3 once Theorem C is proved. The only one which is relevant to Theorem A is:

Corollary C2. *If M is a simple R-module, then either every maximal ideal of S contains some conjugate of Λ, or else there is a maximal ideal of S whose conjugates intersect trivially.*

To see this, suppose that L is a maximal ideal of S which contains no conjugate of Λ. Since F is free, it is distinct from FL, and it follows from Corollary C1 that ML is different from M. We write ${}^0L$ for the intersection of all the conjugates of L under G. Since $x\,{}^0L = {}^0L\,x$ for every x in G, it follows that $M\,{}^0L$ is an R-submodule of M. Since $M\,{}^0L \leqslant ML < M$ and M is simple, we deduce that $M\,{}^0L = 0$. That ${}^0L$ is zero now follows from the assumption that M is torsion-free as an S-module.

In the proof of Theorem A this will be applied with the ring J an absolute field and A infinite. It is a simple matter to show that in these circumstances the intersection of the conjugates of no maximal ideal of S can be trivial.

Lemma 1. *Suppose k is an absolute field, A a normal subgroup of the polycyclic by finite group G and P an ideal of the group algebra S of A over k. If* $\dim_k(S/P)$ *is finite, then some non-trivial power A^m of A lies in $1 + {}^0P$ and* $\dim_k(S/{}^0P)$ *is finite.*

Here, of course, ${}^0P = \bigcap_{x \in G} P^x$. Since k is absolute, any finitely generated subgroup of $\mathrm{GL}_n(k)$ is finite. It follows that the image of A in S/P under the homomorphism $a \to P + a$, $a \in A$, is finite. Hence the kernel $(1 + P) \cap A$ has finite index in A. There therefore exists a positive integer m with $A^m \leqslant 1 + P$. Since A^m is normal in G, we deduce that $A^m \leqslant 1 + {}^0P$. Now A/A^m is finite, so that $S/{}^0P$, as an image of $k(A/A^m)$, is finite-dimensional over k.

If A is Abelian, the Weak Nullstellensatz shows that Lemma 1 is applicable to maximal ideals of S. It follows that *if L is a maximal ideal of S and A is infinite, then ${}^0L$ is not zero.*

This will leave us with the first possibility in Corollary C2, and to dispose of that we consider special Abelian normal subgroups of G.

2.3. *Plinths*

Very little is known about the ideal structure of group algebras of an Abelian group A in relation to a group of operators. To use that little, we need the idea of a plinth. Let G be any group. By a *plinth of G* we shall mean a free Abelian normal subgroup A of finite positive rank which has no non-trivial subgroup of lower rank normal in any subgroup of finite index in G. In other words, G and all its subgroups of finite index must act rationally irreducibly on A.

Our arguments depend ultimately upon the important work of Bergman. We quote the main result of [2] as:

Theorem D. *Suppose that A is a plinth of the group G, that K is any field and that $S = KA$. If P is a non-zero ideal of S normalized by a subgroup of finite index in G, then* $\dim_K(S/P)$ *is finite.*

On the basis of this result we shall prove, in Section 4:

Theorem E. *Suppose that A is a plinth of the polycyclic by finite group G, that k is an absolute field and that $S = kA$. If Λ is any non-zero ideal of S, then there is a maximal ideal of S which contains no conjugate of Λ.*

Together with Corollary C2 and Lemma 1, this shows that *no simple kG-module can be torsion-free for the subalgebra generated by a plinth.* However, infinite polycyclic groups need not have plinths, so that the theorem has to be applied in conjunction with the easy

Lemma 2. *An infinite Abelian normal subgroup of a polycyclic by finite group G contains a plinth of a normal subgroup of finite index in G.*

Proof. If A is an infinite Abelian normal subgroup of G, then there are powers A^m of A which are non-trivial and free. All these are normal in G. Let A_0 be a non-trivial free subgroup of A of least possible rank subject to having only finitely many conjugates in G. If G_0 is the normalizer of A_0 in G, and G_1 is the intersection of the conjugates of G_0 in G, then $A_0 \cap G_1$ is a plinth of G_1, and G_1 is a normal subgroup of finite index in G.

2.4. *R-modules which are not torsion-free for S*

Let S be any ring and M any S-module. For a subset X of S we write $*X$ for the victim of X in M. We shall write $\pi_S(M)$ for the set of all ideals P of S which are maximal with respect to $*P > 0$. If M is non-zero and S has the maximal condition on ideals, then $\pi_S(M)$ is non-empty and consists solely of prime ideals of S. The induction steps in the proofs rest upon the well-known fact that *if $P_1, P_2, \ldots, P_n$ are all different and in $\pi_S(M)$, then $*P_1 + *P_2 + \ldots + *P_n$ is a direct sum*. For P_1 is prime and contains none of $P_2, \ldots, P_n$, so that $P_1 < P_1 + P_2P_3 \ldots P_n$. That $*P_1 \cap (*P_2 + \ldots + *P_n)$ is zero follows from the definition of $\pi_S(M)$ and the fact that it is killed by $P_1 + P_2P_3 \ldots P_n$. With this remark we may prove:

Lemma 3. *Let J be any ring and R the group ring of a group G over J. Let S be the group ring over J of a normal subgroup H of G. Suppose that M is any R-module. If P is in $\pi_S(M)$ and T is a transversal to the cosets of the normalizer of P in G, then*

$$(*P)R = \sum_{t \in T} (*P)t$$

and this sum is direct.

Let N be the normalizer of P in G. For x in G it is clear that $(*P)x = *(P^x)$. Since $G = NT$ and $*P$ is a JN-submodule, it follows that $(*P)R = \Sigma_t(*P)t = \Sigma_t *(P^t)$. If $t_1, \ldots, t_n$ are mutually distinct members of T, then $P^{t_1}, P^{t_2}, \ldots, P^{t_n}$ are all different members of $\pi_S(M)$. We deduce that the sum $*(P^{t_1}) + *(P^{t_2}) + \ldots + *(P^{t_n})$ is direct.

We remark that

(2) if M is simple, then $*P$ is a simple JN-module.

Indeed, if U is any JN-submodule of $*P$, then $UR = \Sigma_t\, Ut$. This sum is direct and, obviously, $U = ((*P)t \cap UR)t^{-1}$. The correspondence $U \to UR$ is therefore one-to-one from the set of JN-submodules of $*P$ to the set of R-submodules of M. Because it clearly preserves inclusions, we may also say, writing *Max-R* for the maximal condition on R-submodules, that

(3) if M has Max-R, then $*P$ has Max-JN.

It is (2) and (3) which allow us to replace consideration of the pair M, JG by that of the pair $*P, JN$.

2.5. *Proof of Theorem A from Theorems C, D and E*

We suppose that G is any polycyclic by finite group, that k is an absolute field and that $R = kG$. If G is finite, then every finitely generated R-module has finite dimension over k. We assume therefore that G is infinite and proceed to prove by induction on the Hirsch number of G that every simple R-module is finite-dimensional over k. We recall that the *Hirsch number* is the number of infinite cyclic factors in any series (1) in which all the factors are either finite or infinite cyclic.

Let M be any simple R-module. Since G is infinite, it has infinite Abelian normal subgroups, so by Lemma 2 there is a normal subgroup G_1 of finite index in G which has a plinth A. Because G_1 has finite index, M is finitely generated as a kG_1-module and it follows that there is a maximal kG_1-submodule U of M. Now $\bigcap_{x \in G} Ux$, being a proper R-submodule of M, is zero; and there are only finitely many distinct Ux. It is sufficient therefore to show that $\dim_k (M/U)$ is finite. In other words, we may assume that G_1 equals G and that U is zero. We shall write S for kA.

From the remark after Theorem E, the module M cannot be torsion-free as an S-module. The considerations of the previous paragraph can therefore be applied. Suppose P is in $\pi_S(M)$ and let N be the normalizer of P in G. From (2), $^*P$ is a simple kN-module. If N has infinite index in G, then it has a lower Hirsch number than G. We may deduce from the induction hypothesis that $\dim_k(^*P)$ is finite. Since S/P is faithfully represented by endomorphisms of $^*P$, it follows that $\dim_k(S/P)$ is finite. If N has finite index in G, the same conclusion holds, but by Theorem D.

Lemma 1 now shows that A^m is in $1 + {}^0P$ for some non-zero m. Since ^0P kills every $(^*P)x$, it kills M. It follows that A^m acts trivially on M, and therefore M is a $k(G/A^m)$-module. Since G/A^m has a lower Hirsch number than G, we deduce that $\dim_k (M)$ is finite.

2.6. *Proof of Corollary A*

All we need do is substantiate the remark made in the introduction. This we do with:

Corollary C3. *Suppose that J is a commutative Hilbert ring, G is a polycyclic by finite group and R is the group ring JG. If M is a simple R-module, then M is killed by a maximal ideal of J.*

Proof. Let P be the killer of M in J. Since victims in M of elements of J are R-submodules, M is torsion-free as a J/P-module. To show that J/P is a field we may assume that P is zero. Theorem C may now be applied with $A = 1$. The conclusion of Corollary C2 is that either Λ lies in every maximal ideal of J, or else there is some maximal ideal of J which is zero. However, J is a Hilbert ring and also, because it has M as a torsion-free module, a domain. Therefore the intersection of the maximal ideals is trivial. Since Λ is non-zero, it follows that J is a field, as required.

A more complicated version of this corollary will be needed when we come to Theorem B.

3. Theorem C – Proof and further consequences

3.1. *A more general result*

Because of an induction step in the proof of Theorem C, we must not restrict ourselves to group rings $S = JA$ of Abelian normal subgroups A of G, but consider slightly more general situations. *We shall suppose throughout this section that R is a ring generated by a subring S and a group G, and that G normalizes S* in the sense that $\sigma^x = x^{-1}\sigma x$ is in S for every σ in S and x in G.

Suppose that U is an S-module. If x is in G, we may take an additive group U^x isomorphic with U via an isomorphism $u \to u^x$, $u \in U$, and convert it into an S-module by defining

$$u^x \sigma = (u\,\sigma^{x^{-1}})^x$$

for all u in U and σ in S. A trivial verification shows that $U \cong U^1$, that $U^x \cong V^x$ if $U \cong V$, and that $(U^x)^y \cong U^{xy}$ if y is a further element of G. It follows that G permutes the isomorphism types of S-modules. For S-modules U and V, we shall write $U \sim_G V$ to mean that U and V are *G-conjugate*, that is to say $V \cong U^x$ for some x in G. This concept helps in analyzing R-modules in terms of their structure as S-modules because of the

Cancellation Law. *Let M be any R-module and suppose $V \leqslant U$ are S-submodules. If x is in G, then*

$$Ux/Vx \sim_G U/V.$$

This holds simply because $(V+u)x\sigma = (V+u)\sigma^{x^{-1}}x$ for every u in U and σ in S. With its help we may prove:

Theorem C*. *Suppose S has Max-r and G is polycyclic by finite. If M is any finitely generated R-module, then there is an ascending series*

$$0 = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_\lambda \leqslant M_{\lambda+1} \leqslant \ldots \leqslant M_\rho = M \tag{4}$$

of S-submodules such that the factors $M_{\lambda+1}/M_\lambda$, $0 \leqslant \lambda < \rho$, are all cyclic and fall into finitely many conjugacy classes under G.

Of course, to say that (4) is a series means that if μ is a non-zero limit ordinal no greater than ρ, then $M_\mu = \bigcup_{\lambda<\mu} M_\lambda$.

It will help in proving this theorem, which has Theorem C as an easy consequence, to have some more notation. Let $\mathfrak{X}$ be any class of S-modules. We shall write $\mathrm{P}\mathfrak{X}$ for the class of all poly-$\mathfrak{X}$ modules; so that an S-module M is in P if and only if there is a chain $0 = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_n = M$ of finitely many S-submodules such that the factors M_{i+1}/M_i, $0 \leqslant i < n$, are all in the class $\mathfrak{X}$. For example, if $\mathfrak{X}$ is the class of all cyclic S-modules, then P $\mathfrak{X}$ is the class of all finitely generated S-modules. These could, if one had a mind for it, be called the polycyclic S-modules. The operator P is a closure operator in the sense that

$$\mathfrak{X} \leqslant \mathrm{P}\mathfrak{X} = \mathrm{P}^2\mathfrak{X} \leqslant \mathrm{P}\mathfrak{Y}$$

whenever $\mathfrak{X} \leqslant \mathfrak{Y}$; and $\mathrm{P}(0) = (0)$.

It is the transfinite version of P which is more relevant to Theorem C*. We define $\acute{\mathrm{P}}$ by saying that M is in $\acute{\mathrm{P}}\mathfrak{X}$ if and only if there is an ascending series (4) of S-submodules such that each of the $M_{\lambda+1}/M_\lambda$, $0 \leqslant \lambda < \rho$, is in $\mathfrak{X}$. The operators P and $\acute{\mathrm{P}}$ have been familiar in group theory since their introduction by Hall, as E and $\acute{\mathrm{E}}$, in [7].

If we write $\mathfrak{X}^G$ for the class of all S-modules which are G-conjugate to a member of $\mathfrak{X}$, then the conclusion of Theorem C* could be put as

$$M \in \acute{\mathrm{P}}(\mathfrak{X}^G) \tag{5}$$

for some finite class $\mathfrak{X}$ of cyclic S-modules.

Of course $\mathfrak{X} \to \mathfrak{X}^G$ is also a closure operation. Its relationship with P and $\acute{\mathrm{P}}$ is expressed in

$$(\mathrm{P}\mathfrak{X})^G \leqslant \mathrm{P}(\mathfrak{X}^G), \qquad (\acute{\mathrm{P}}\mathfrak{X})^G \leqslant \acute{\mathrm{P}}(\mathfrak{X}^G). \tag{6}$$

3.2. *Proof of Theorem C**

We consider first the case when $G = \langle x \rangle$ is infinite cyclic. The ring R is then $\sum_{n=-\infty}^{+\infty} Sx^n$. Suppose that U is a finitely generated S-submodule of M with $M = UR$. Obviously $M = \sum_n Ux^n$. We shall construct a series (4) with $\rho = \omega 2$. Let

$$U_0^+ = U_0^- = 0,$$

$$U_n^+ = Ux + Ux^2 + \ldots + Ux^n, \qquad n > 0, \tag{7}$$

$$U_n^- = Ux^{-1} + Ux^{-2} + \ldots + Ux^{-n}, \qquad n > 0. \tag{8}$$

Clearly

$$0 = U_0^- \leqslant U_1^- \leqslant \ldots \leqslant U_n^- \leqslant \ldots . \tag{9}$$

This will form half of the series. Since $U_{n+1}^- x = U + U_n^-$, we may write

$$V = \bigcup_{n=0}^{\infty} U_n^-, \tag{10}$$

and be sure that

(11) $V \leqslant Vx \leqslant Vx^2 \leqslant \ldots \leqslant Vx^n \leqslant \ldots$.

This will be the second half, since obviously $M = \bigcup_{n=0}^{\infty} Vx^n$.

The factors Vx^{n+1}/Vx^n, $n \geqslant 0$, of the chain (11) are all G-conjugate, by the Cancellation Law, to Vx/V. But $Vx = U + V$, so that Vx/V is isomorphic with $U/(U \cap V)$. This is a finitely generated S-*module and therefore lies in* $\mathrm{P}\mathfrak{X}_0$ for some finite class $\mathfrak{X}_0$ of cyclic S-modules. Hence

(12) $M/V \in \acute{\mathrm{P}}(\mathfrak{X}_0^G)$.

We now consider the factors of the chain (9). For $n \geqslant 0$, we have $U_{n+1}^- = U_n^- + Ux^{-(n+1)}$. Hence U_{n+1}^-/U_n^- is isomorphic with $Ux^{-(n+1)}/(U_n^- \cap Ux^{-(n+1)})$. By the Cancellation Law this is G-conjugate to $U/(U \cap U_n^- x^{n+1})$. However, (7) and (8) show that $U_n^- x^{n+1} = U_n^+$. It follows that

(13) $U_{n+1}^-/U_n^- \sim_G U/(U \cap U_n^+), \qquad n \geqslant 0.$

Now S has Max-r and U is a finitely generated S-module. The ascending chain

$$0 \leqslant U \cap U_1^+ \leqslant U \cap U_2^+ \leqslant \ldots \leqslant U \cap U_n^+ \leqslant \ldots$$

of submodules therefore becomes stationary. Suppose $U \cap U_N^+$ is the limit. Since $U/(U \cap U_i^+)$, $0 \leqslant i \leqslant N$, are finitely many polycyclic S-modules, there is a finite class $\mathfrak{X}_1$ of cyclic S-modules such that each of them lies in $\mathrm{P}\mathfrak{X}_1$. From (10) and (13) it follows that V is in $\acute{\mathrm{P}}(\mathrm{P}\mathfrak{X}_1)^G$. From (6) we deduce that V is in $\acute{\mathrm{P}}(\mathfrak{X}_1^G)$. Combining this with (12), we may put $\mathfrak{X} = \mathfrak{X}_0 \cup \mathfrak{X}_1$ to give (5) in this case.

The theorem in general follows from the particular case. Since G is polycyclic by finite there is a series

$$1 = G_n \lhd G_{n-1} \lhd \ldots \lhd G_1 \lhd G_0 = G,$$

where the factors are either finite or infinite cyclic. We proceed by induction on n. Let S_1 be the subring of R generated by S and G_1. If G/G_1 is finite, then M is finitely generated as an S_1-module, and we may deduce from the induction hypothesis that M lies in $\acute{\mathrm{P}}(\mathfrak{X}^{G_1})$, and hence in $\acute{\mathrm{P}}(\mathfrak{X}^G)$ for a suitable class $\mathfrak{X}$. We suppose therefore that G/G_1 is infinite cyclic, so that there is an infinite cyclic subgroup H of G with $G = G_1H$. Obviously R is generated by S_1 and H, and H normalizes S_1. Moreover, S_1 has Max-r by the methods of Hall [4]. From the infinite cyclic case already established we deduce that M, as an S_1-module, lies in $\acute{\mathrm{P}}(\mathfrak{Y}^H)$ for some finite class $\mathfrak{Y}$ of cyclic S_1-modules.

Let $\mathfrak{Y}$ be determined by the S_1-modules $Y_1, \ldots, Y_m$. By induction, each Y_i, as an S-module, lies in $\acute{\mathrm{P}}(\mathfrak{X}_i^{G_1})$ for some finite class $\mathfrak{X}_i$ of cyclic S-modules. If we put $\mathfrak{X} = \bigcup_{i=1}^m \mathfrak{X}_i$, then $\mathfrak{Y}$, as a class of S-modules, lies in $\mathfrak{Z} = \acute{\mathrm{P}}(\mathfrak{X}^{G_1})$. It follows that M, as an S-module, lies in $\acute{\mathrm{P}}(\mathfrak{Z}^H)$. Since $\mathfrak{X}^{G_1}$ and $\mathfrak{X}^H$ are both subclasses of $\mathfrak{X}^G$, we deduce from (6) that M is in $\acute{\mathrm{P}}(\mathfrak{X}^G)$.

3.3. *Proof of Theorem C*

Here we suppose that S is commutative and Noetherian and that M is a finitely generated R-module which is torsion-free as an S-module. The group G is supposed to be polycyclic by finite. We show that there is a free S-submodule F of M and a non-zero ideal Λ of S such that every finitely generated S-submodule of M/F is killed by a product $\Lambda^{x_1}, \Lambda^{x_2} \dots \Lambda^{x_n}$ of conjugates of Λ under G. Theorem C follows from specializing S.

According to Theorem C*, there are finitely many cyclic S-modules $U_1, \dots, U_r$, which we may suppose to be mutually inconjugate under G, and an ascending series (4) such that each of the factors $M_{\lambda+1}/M_\lambda$ is G-conjugate to one of $U_1, \dots, U_r$. Obviously we may suppose that M_1 is non-zero and isomorphic with U_1. Since M is torsion-free as an S-module, we may take $U_1 = S$.

Let $\mathcal{O}$ be the set of all λ, $0 \leqslant \lambda < \rho$, such that $M_{\lambda+1}/M_\lambda$ is isomorphic with S, and suppose that $M_{\lambda+1} = M_\lambda + \mu_\lambda S$ for each λ. The S-submodule

$$F = \sum_{\lambda \in \mathcal{O}} \mu_\lambda S$$

is then free. If $r = 1$, then $F = M$, and we may take Λ to be S. If $r > 1$, we show that we may take

$$\Lambda = P_2 P_3 \dots P_r$$

to be the product of the killers P_i of U_i in S.

Now U_i is not isomorphic with S for $i \geqslant 2$, so that P_i is non-zero. Since S is a domain, it follows that Λ is non-zero. All that remains to show is that if U is a finitely generated S-submodule of M, then

$$U \Lambda^{x_1} \Lambda^{x_2} \dots \Lambda^{x_n} \leqslant F \tag{14}$$

for some $x_1, x_2, \dots, x_n$ in F.

Let μ_1 be the least ordinal such that U is contained in $F + M_{\mu_1}$. Since U is finitely generated, μ_1 is either zero or else a successor ordinal $\lambda_1 + 1$. Hence if U is not contained in F, then $\mu_1 = \lambda_1 + 1$. Now $F + M_\lambda = F + M_{\lambda+1}$ if λ is in $\mathcal{O}$; therefore λ_1 is not in $\mathcal{O}$. It follows that M_{μ_1}/M_{λ_1} is G-conjugate to one of $U_2, \dots, U_r$ and consequently killed by a conjugate of one of $P_2, \dots, P_r$. We deduce that there is some x_1 in G with $M_{\mu_1}\Lambda^{x_1} \leqslant M_{\lambda_1}$. Therefore $U\Lambda^{x_1} \leqslant F + M_{\lambda_1}$. Since S is Noetherian, $U\Lambda^{x_1}$ is also a finitely generated S-submodule of M, and if μ_2 is the least ordinal such that $U\Lambda^{x_1} \leqslant F + M_{\mu_2}$, then $\mu_2 < \mu_1$. (14) follows by induction.

3.4. *Further remarks, when S is central in R*

Because of our scant knowledge about groups acting on group algebras, there are difficulties in applying Theorems C and C* to modules which are not torsion-free for S. If S is in the centre of R, these disappear. Recalling that an S-module is said to be *prime* if and only if it is non-zero and is killed by the killer of any non-zero submodule, we state

Corollary C4. *Suppose S is a Hilbert ring in the centre of R and G is polycyclic by finite. If M is a finitely generated R-module which is prime as an S-module, then*

$$\bigcap_{L \lhd | S} ML = 0.$$

Here, and henceforth, the symbol $\lhd|$ will mean 'is a maximal ideal of'.

Proof. Let P be the killer of M in S, and let R_1 be the group ring of G over $S_1 = S/P$. Since R/PR is an image of R_1, we may view M as a finitely generated R_1-module. Because M is prime for S, it is torsion-free for S_1. From Theorem C and Corollary C1, with $A = 1$, there is a free S_1-submodule F of M and a non-zero ideal Λ such that

(15) if $\Lambda \not\leqslant L \lhd| S$, then $(ML) \cap F = FL$.

Let $\mathfrak{L}$ be the set of maximal ideals of S_1 which do not contain Λ, and write $\Delta = \bigcap_{L \in \mathfrak{L}} L$. The ideal $\Lambda\Delta$ of S_1 is in every maximal ideal and is therefore zero. It follows that Δ is zero. Let X be $\bigcap_{L \in \mathfrak{L}} ML$. From (15) we deduce that $X \cap F$ equals $\bigcap_{L \in \mathfrak{L}} FL$. Since F is free, this is the same as $F\Delta$, that is to say zero. Hence X is isomorphic with $(F + X)/F$. However, M/F is a torsion S_1-module and X is torsion-free; therefore $X = 0$.

This result corresponds to [5, Lemma 12]. It may be extended to arbitrary finitely generated R-modules by remarking that if M is any non-zero finitely generated R-module, then there is a chain

$$0 = M_0 < M_1 < \ldots < M_n = M$$

of R-submodules of M such that

$$M_{i+1}/M_i \text{ is prime for } S$$

for each i, $0 \leqslant i < n$. For if P is in $\pi_S(M)$, then $^*P$ is a prime S-submodule and, because S is in the centre of R, an R-submodule. We may take $M_1 = {}^*P$ and consider M/M_1. The remark follows since M has Max-R.

If we write T_S for the operator which assigns to each S-module U the submodule

$$\mathrm{T}_S(U) = \bigcap_{L \lhd S} UL,$$

then Corollary C4 shows that $\mathrm{T}_S(M_{i+1}) \leqslant M_i$ for $0 \leqslant i < n$. We conclude that

(16) $\mathrm{T}_S^n(M) = 0$.

Other results may be deduced from this, particularly if account is taken of the Artin–Rees Lemma [16, p. 208]: if P is an ideal of S and U is an R-submodule of M, then there is some m with $MP^m \cap U \leqslant UP$. For example *if λ is an element of S which kills every simple R-image of M, then* $M\lambda^n = 0$. To see this, all that is necessary is to show that if U is any submodule of M, then $U\lambda \leqslant \Gamma_S(U)$, for then $\Gamma_S^i(M)\lambda \leqslant \Gamma_S^{i+1}(M)$, and appeal may be made to (16). However, if $L \lhd| S$ and $U\lambda \nleqslant UL$, then λ is not in L. It follows that no simple image of M can be killed by L, and hence $M = ML$. From the Artin–Rees Lemma, we deduce that $U = UL$, so that $U\lambda \leqslant UL$ after all. An easy consequence of this is:

Corollary C5. *Suppose J is a commutative Hilbert ring and G is polycyclic by finite. Suppose M is a finitely generated JG-module and ϕ an endomorphism of it. If $M\phi$ is contained in every maximal ϕ-invariant JG-submodule of M, then $M\phi^n = 0$ for some n.*

For if ψ is the natural homomorphism of JG into the endomorphism ring E of M, we may take R to be the subring of E generated by $(JG)^\psi$ and ϕ, and S to be that generated by J^ψ and ϕ. If we view M as an R-module, the preceding arguments, with $\lambda = \phi$ and G^ψ in place of G, can be applied since S, as an image of $J[X]$, is a Hilbert ring. We deduce that $M\phi^n = 0$.

It is a pleasure to record thanks to K.W. Gruenberg for bringing this result to our attention. The remarks in the introduction about JG being Hilbert come directly from it. Suppose that X is an ideal of JG and that M is JG/X. Let $X + \lambda$ be an element of the Jacobson radical of JG/X. The mapping $\phi : X + r \to X + \lambda r$, $r \in JG$, is an endomorphism of M. Suppose U is a maximal ϕ-invariant JG-submodule of M. If $M\phi$ is not contained in U, then $M = U + M\phi$, so that $M\phi$ is not contained in any proper JG-submodule of M containing U. This contradicts the fact that $X + \lambda$ is in the Jacobson radical of JG/X. It follows from Corollary C5 that $M\phi^n = 0$ for some n, and hence λ^n lies in X. The Jacobson radical of JG/X is therefore a nil ideal and must be nilpotent by the theorem of Levitzki [12].

There are other ways of proving Corollary C5 which are more economical, but the theory of central ideals is well enough known for us not to pursue them.

4. Theorem E – Proof

4.1. *Reductions*

Suppose that k is an absolute field, that A is a plinth of the polycyclic by finite group G and that $S = kA$. Let $\lambda = \Sigma\lambda_a a$ be a non-zero element of S. We must show that some maximal ideal of S contains no conjugate of λ.

Let k_0 be the subfield of k generated by the coefficients λ_a of λ. Since k is absolute, k_0 is finite. Suppose that $S_0 = k_0A$ and that L_0 is a maximal ideal of S_0 which contains no conjugate of λ. Since S is free over S_0 there is a maximal ideal L of S containing L_0. Obviously $S \cap L = S_0$, so that L contains no conjugate of λ either. We may therefore assume that $k = k_0$ is a finite field.

By [13, Theorem 1], there is a normal subgroup G_0 of finite index in G which contains A and induces an Abelian group of automorphisms of A. Clearly A is a plinth of G_0. Let T be a transversal to the cosets of G_0 in G and $\lambda_0 = \Pi_{t \in T} \lambda^t$. Since S is a domain, λ_0 is non-zero. If L is a maximal ideal of S which contains no conjugate of λ_0 under G_0, then L can contain no conjugate of λ under G. We may therefore assume that $G = G_0$.

Suppose that we have proved that there is an element x of G such that

(17) A is a plinth of $\langle A, x\rangle$,

(18) x normalizes infinitely many maximal ideals of S.

Let $\mathfrak{L}$ be the set of all maximal ideals of S which are normalized by x. If L is in $\mathfrak{L}$ then, because G induces an Abelian group of automorphisms of S, so are all the conjugates of L under G; and Lemma 1 shows that there are only finitely many of these. Hence, and from (18), $\mathfrak{L}$ breaks up into infinitely many conjugacy classes. If every member of $\mathfrak{L}$ contained a conjugate of λ, then there would exist infinitely many members of $\mathfrak{L}$ which contained λ, and with it the ideal P of S generated by all the conjugates of λ under $\langle x\rangle$. However, Theorem D and (17) show that $\dim_{k}(S/P)$ is finite, so that P is contained in only finitely many maximal ideals of S. Theorem E therefore follows from (17) and (18).

4.2. *Proof of (17)*

We appeal to an unpublished result of D.S. Passman, whose kind permission to use it here we gratefully acknowledge.

Lemma 4. *Suppose Γ is a finitely generated Abelian group and V a $\mathbf{Q}\Gamma$-module. If V is a simple $\mathbf{Q}\Gamma_0$-module for every subgroup Γ_0 of finite index in Γ, then there is an element ξ of Γ such that V is a simple $\mathbf{Q}\langle\xi^m\rangle$-module for every positive m.*

To apply it we take Γ to be the group of automorphisms of A induced by G and view A as a $\mathbf{Z}\Gamma$-module. The hypotheses of the lemma, with $V = A \otimes \mathbf{Q}\Gamma$ are a simple translation of the fact that A is a plinth of G. For (17) we may take x to be any element of G which induces ξ.

We prove Lemma 4. We may obviously suppose that Γ does not act trivially on V. There is a maximal ideal L of $\mathbf{Q}\Gamma$ with V isomorphic with $\mathbf{Q}\Gamma/L$. We set F equal to $\mathbf{Q}\Gamma/L$, and remark that F is a finite extension of $\mathbf{Q}$ by the Weak Nullstellensatz. There are therefore only finitely many proper subfield of F, which we shall call $F_1, F_2, \ldots, F_n$.

For $1 \leqslant i < n$, let X_i/L be F_i and set $\Gamma_i = X_i \cap \Gamma$. We write Δ_i/Γ_i for the periodic part of Γ/Γ_i. Each Γ/Δ_i is torsion-free and, because Γ is finitely generated, each Δ_i/Γ_i is finite. Now F_i corresponds to a proper non-zero $\mathbf{Q}\Gamma_i$-submodule of V, so that from the hypothesis Γ_i must have infinite index in Γ. It follows that each Δ_i has infinite index in Γ. A result of Neumann [14], or a simple direct argument, now shows that there is an element ξ of Γ which is contained in none of $\Delta_1, \Delta_2, \ldots, \Delta_n$. This element ξ has no positive power in any of the Γ_i and hence no positive power in any of the X_i.

Now let $\eta = L + \xi$, and suppose that m is positive. Obviously η^m cannot lie in any proper subfield of F. It follows that $\mathbf{Q}\Gamma = L + \mathbf{Q}\langle \xi^m \rangle$, so that V is a simple $\mathbf{Q}\langle \xi^m \rangle$-module, as required.

4.3. *Proof of (18)*

We remark that the truth of (18) does not depend on (17): it is true if x is replaced by any automorphism of A, and follows obviously from:

Lemma 5. *Suppose k is a finite field and S is the group algebra of a free Abelian group A over k. If ξ is an automorphism of A, then the intersection of the maximal ideals of S which are normalized by ξ is zero.*

Proof. We suppose that A is non-trivial and that k has order q. If $L = L^\xi \lhd S$, then ξ induces a k-automorphism of the finite extension S/L of k. Hence there exists some $n \geqslant 0$ such that $L + \sigma^\xi = L + \sigma^{q^n}$ for every σ in S. It follows that $a^\xi a^{-q^n} - 1$ lies in L for every a in A. If we write

$$A_n = \langle a^\xi a^{-q^n} ; a \in A \rangle,$$

and

$$\bar{\mathfrak{a}}_n = \sum_{\alpha \in A_n} (\alpha - 1)S$$

for the kernel of the natural homomorphism of S onto $k(A/A_n)$, then

$$\bar{\mathfrak{a}}_n \leqslant L.$$

Conversely, if $\bar{\mathfrak{a}}_n \leqslant L \lhd S$, then $L = L^\xi$. For if f is the Frobenius endomorphism $\sigma \to \sigma^q$ of S, then ξ acts like f^n in A/A_n, and it follows that ξ acts like f^n in $S/\bar{\mathfrak{a}}_n$. However, f stabilizes every ideal of S.

Now the Jacobson radical of $S/\bar{\mathfrak{a}}_n$ is normalized by ξ and must therefore be idempotent. Since it is also nilpotent, it must be trivial. Therefore the intersection of the maximal ideals of S which are normalized by ξ is the same as $\bigcap_{n=0}^\infty \bar{\mathfrak{a}}_n$. To prove the lemma, we must show that this is zero.

As a matter of fact, if I is any infinite subset of the natural numbers, then $\bigcap_{n \in I} \mathfrak{a}_n$ is zero. This will follow from proving the corresponding group-theoretic statement

$$(19) \qquad \bigcap_{n \in I} A_n = 1.$$

For suppose this is true and $\lambda = \Sigma\lambda_a a$ is a non-zero element of S. From (19), no non-unit element of A can be contained in infinitely many A_n. Since the set

$$\{ab^{-1} : a \neq b, \lambda_a \lambda_b \neq 0\}$$

is finite, there exists N such that no element of it lies in any A_n with $n \geqslant N$. It follows that λ does not belong to $\mathfrak{a}_n$ if $n \geqslant N$.

To prove (19), we use induction on the rank of A. Suppose that there is a subgroup $B = B^{\xi}$ with $1 < B < A$ and A/B torsion-free. By induction, $\bigcap_{n \in I}(A/B)_n$ is trivial, and therefore $\bigcap_{n \in I} A_n$ is contained in B. Also by induction $\bigcap_{n \in I} B_n$ is trivial. However, it is easily seen that B_n equals $A_n \cap B$, so that $\bigcap_{n \in I} A_n = 1$.

It remains to consider the case when no such B exists. Let C be $\bigcap_{n \in I} A_n$. If $\chi(t)$ is the characteristic polynomial of ξ, then $|A/A_n|$ equals $|\chi(q^n)|$. Since $|\chi(q^n)|$ tends to infinity with n, and since I is infinite, the orders $|A/A_n|$ with n in I are unbounded. Therefore A/C is infinite, and if B/C is the periodic part of A/C, then A/B is infinite as well. Since $C = C^{\xi}$, it follows that $B = B^{\xi}$. Therefore B, and hence C as well, is trivial.

We do not know if ξ can be replaced in Lemma 5 by an arbitrary finitely generated Abelian group of automorphisms and we leave this question open. If it could, then of course there would be no need to introduce the element x. Nor do we know how far the hypothesis that A is a plinth is necessary for Theorem E. If A is free and G/A is cyclic, then Theorem E holds whether or not A is a plinth, but we do not prove that here.

5. Theorem B – Proof

5.1. *Reductions*

Suppose J is a commutative Hilbert ring all of whose capitals are absolute, and that H is a nilpotent normal subgroup of the polycyclic by finite group G. Let M be a finitely generated non-zero JG-module and $\mathfrak{h}$ the augmentation ideal of JH. The hypothesis is that $\mathfrak{h}$ kills every simple image of M, and we must show that $M\mathfrak{h}^n = 0$ for some n.

The first point to make is that if H is contained in a normal subgroup G_0 of finite index in G, then $\mathfrak{h}$ kills every simple JG_0-image of M. For let U be a maximal JG_0-submodule of M and let $V = \bigcap_{x \in G} Ux$. Since G/G_0 is finite, M/V has a finite series of JG_0-submodules the factors of which are each isomorphic with a G-conjugate of M/U. Let W be a maximal JG-submodule of M containing V, and let U_1 be a maximal JG_0-submodule of M containing W. Since $\mathfrak{h}$ kills M/W, it kills M/U_1. Now M/U_1 is isomorphic, as a JG_0-module, with some M/Ux with x in G, and H is normal in G. It follows therefore that $M\mathfrak{h} \leqslant U$.

Now we use induction on the Hirsch number of G and the class of H, and show that we may assume that H is a plinth of G.

If H is finite, we take A to be its centre and let G_0 be the centralizer of A in G. The ring JA is in the centre of JG_0. From the remarks preceding Corollary C5, every element of the augmentation $\mathfrak{a}$ of JA acts nilpotently on M. Since $\mathfrak{a}$ is finitely generated, it follows that $M\mathfrak{a}^{n} = 0$ for some n. Now $M/M\mathfrak{a}$ is a $J(G/A)$-module, and H/A has lower class than H. We deduce that $M\mathfrak{b}^{m} \leqslant M\mathfrak{a}$ for some m. Therefore $M\mathfrak{b}^{mn} = 0$.

If H is infinite, then by [5, Lemma 7] its centre is also infinite, and, of course, normal in G. By Lemma 2 it contains a plinth A of some normal subgroup G_0 of finite index in G. If we show that $M\mathfrak{a}^{n} = 0$, then, because G/A has lower Hirsch number than G, we shall conclude, as before, that $M\mathfrak{b}^{mn} = 0$ for some m. In proving that $\mathfrak{a}$ acts nilpotently on M, we may obviously assume that $G = G_0$. In other words, we may assume that $H = A$ is a plinth of G.

5.2. *The plinth case*

Here we write R for JG and S for JA. Let M_0 be an R-submodule maximal with respect to containing none of the submodules $M\mathfrak{a}^{r}$, $r = 0, 1, 2, \ldots$. By replacing M with M/M_0, if necessary, we may assume that $M_0 = 0$. Hence if V is a non-zero R-submodule of M, then

(20) $\qquad M\mathfrak{a}^{r} \leqslant V$

for some $r = r(V)$. This allows us to assume that

(21) $\qquad M$ is torsion-free as a J-module.

For let P be in $\pi_J(M)$ and U equal $^{*}P$. From the Artin–Rees Lemma there is some n with $MP^{n} \cap U = 0$. If $V = MP^{n}$ is non-zero, then $M\mathfrak{a}^{m} = 0$ if m is the greater of $r(U)$ and $r(V)$. Therefore we may assume that MP^{n} is zero. If MP is not zero, then it contains some $M\mathfrak{a}^{r}$, and $M\mathfrak{a}^{rn} = 0$ follows. Hence we may assume that $MP = 0$. Assumption (21) follows from replacing J with J/P. We remark that now J is a domain.

Next we need

(22) $\qquad$ if $\mathfrak{a} \nleqslant L \lhd\!\!| \; S$, then $M = M\,{}^{0}L$.

For L contains a maximal ideal of J, and the capitals of J are all absolute, so that the group of automorphisms of $S/{}^{0}L$ induced by G is finite by Lemma 1. Hence L has only finitely many conjugates under G. It follows that $S = \mathfrak{a} + {}^{0}L$ and therefore (22) holds.

Suppose that M is torsion-free as an S-module, and consider the non-zero ideal Λ of S associated with M by Theorem C. By (22) and Corollary C1, every maximal ideal of S which does not contain $\mathfrak{a}$ must contain a conjugate of Λ. It follows that every maximal ideal of S contains a conjugate of $\mathfrak{a}\Lambda$. If $P \lhd\!\!| \; J$, then Theorem E, with J/P for k shows that $\mathfrak{a}\Lambda$ lies in PS. Therefore $M\mathfrak{a}\Lambda \leqslant \bigcap_{P \lhd\!| J} MP$. From (21), and Corollary C4 with $S = J$, we deduce that $M\mathfrak{a}\Lambda = 0$, which is a contradiction since both $\mathfrak{a}$ and Λ are non-zero. Hence

(23) $\qquad M$ is not torsion-free as an S-module.

Now suppose that P is in $\pi_S(M)$. (23) shows that P is non-zero and (21) that $P \cap J = 0$. Let $U = {}^*P$ and V equal UR. Suppose N is the normalizer of P in G. We need that J, A, N, U satisfy the same hypotheses as J, H, G, M.

(3) shows that U is a finitely generated JN-module. We must show that $\mathfrak{a}$ kills every simple JN-image of U. Let U_1 be a maximal JN-submodule of U and $V_1 = U_1R$. By Lemma 3, V_1 is a proper submodule of V, so there exists a maximal R-submodule W of V containing V_1. Now V/W is finite-dimensional over some capital of J by Corollary A and is therefore killed by ${}^0L$ for some $L \lhd| S$; moreover, V contains some $M\mathfrak{a}^r$ by (20). From (22), therefore,

$$M\mathfrak{a}^{r+1} = M\mathfrak{a}\,\mathfrak{a}^r \leq M\,{}^0L\,\mathfrak{a}^r = M\mathfrak{a}^r\,{}^0L \leq W.$$

It follows that $V\mathfrak{a} \leq W$, so that $U\mathfrak{a} \leq W \cap U$. Since $W \cap U = U_1$, we deduce that $\mathfrak{a}$ kills U/U_1, as required.

If N has infinite index in G, then from the inductive hypothesis on the Hirsch number of G we may deduce that $U\mathfrak{a}^m = 0$. It follows that $V\mathfrak{a}^m = 0$ and from (20) that $M\mathfrak{a}^{r+m} = 0$.

We therefore assume that N has finite index in G. If J is a field, then Theorem D applies and gives $\dim_J(S/P)$ finite. Lemma 1 now shows that some non-trivial power A^l of A acts trivially on V. Since G/A^l has lower Hirsch number than G, it follows that $V\mathfrak{a}^m = 0$ for some m, and as before, $M\mathfrak{a}^{r+m} = 0$. Hence we may assume that J is not a field.

Theorem D cannot now be directly applied, but an indirect application will allow us to complete the proof. We consider $Y = S/P$ as a J-module and apply Theorem C, with $A = G$ and Y for M. Let Λ be the non-zero ideal of J and F the free J-submodule of Y coming from that theorem. Let $\hat{J}$ be the field of fractions of J and $\hat{S} = \hat{J}A$. If $\hat{P} = P\hat{S}$, then Theorem D shows that $\dim_{\hat{J}}(\hat{S}/\hat{P})$ is finite. It follows at once that the rank, m, of F is finite.

Now Corollary C1 shows that $Y = F + YL$ and $(YL) \cap F = FL$ whenever L is a maximal ideal of J not containing Λ. It follows that Y/YL is isomorphic with F/FL. Hence

(24) if $\Lambda \nleq L \lhd| J$, then $\dim_{J/L}(S/(P + SL)) = m$.

The killer X_L of U/UL in S contains $P + SL$ obviously. If $\Lambda \nleq L \lhd| J$, then (24) shows that the dimension of S/X_L over J/L is at most m. From (21), and since J is not a field, VL is non-zero. From (20), some power of $\mathfrak{a}$ therefore kills U/UL and hence lies in X_L; clearly $\mathfrak{a}^m$ must then lie in X_L. It follows that

$$U\mathfrak{a}^m\Lambda \leq \bigcap_{L \lhd| J} UL.$$

Corollary C4, with $S = J$ and $U = M$, now shows that $\bigcap_{L \lhd| J} UL$ is zero; and (21) that $U\mathfrak{a}^m = 0$. Once again it follows that $V\mathfrak{a}^m = 0$, and the proof is complete.

5.3. *Proof of Corollary B*

This is a very simple matter compared with Theorem B. Here we suppose that H is any normal subgroup of the polycyclic by finite group G and that M is a non-zero finitely generated JG-module. The ring J is again supposed to be a commutative Hilbert ring, all of whose capitals are absolute, and $\mathfrak{h}$ to kill every chief factor of M. We show that some power of $\mathfrak{h}$ kills M.

To do this, we use induction on the Hirsch number of G and, as before, replace M by a suitable image so that if V is a non-zero submodule, then V contains $M\mathfrak{h}^r$ for some r.

If H is infinite, then it has an infinite Abelian subgroup A normal in G. By Theorem B, $M\mathfrak{a}^n = 0$ for some n. Let V be the victim of $\mathfrak{a}$ in M. Obviously V is a $J(G/A)$-module and we deduce that $V\mathfrak{h}^l = 0$ for some l. The result follows.

If H is finite, we may use induction on the order of H. Let A be a minimal normal subgroup of G contained in H. Provided that A is a proper subgroup of H, we deduce that $M\mathfrak{a}^n = 0$ for some n. Since $|H/A| < |H|$, the result follows as above. If $A = H$ is Abelian, the result follows from the Theorem. The only other possibility is that H is equal to its derived subgroup. In this case, $\mathfrak{h} = \mathfrak{h}^2$. It follows that $M\mathfrak{h}$ is contained in every non-zero submodule of M. Therefore $M\mathfrak{h}$ is either zero or simple. In either case, $M\mathfrak{h}^2 = 0$.

5.4. *Further remarks*

To show that the nilpotency hypothesis is necessary in Theorem B we now suppose that H is a non-nilpotent normal subgroup of the polycyclic by finite group G, and show that there is a finite prime field $\mathscr{k}$ and a finite $\mathscr{k}G$-module M such that H kills every simple image of M and yet fails to act nilpotently.

It follows from the work of Hirsch [8] that there is a homomorphism θ of G onto a finite group such that H^θ is not nilpotent. By replacing G and H by their images, we may assume that G is finite. Let B be the limit of the lower central series of H, and p a prime dividing the order of B. We consider $R = \mathscr{k}G$. Obviously, if $\in = \Sigma_{h\in H}\, h$, then

$$\epsilon = \Sigma_{h\in H} h, \text{ then}$$

$$(25) \qquad \epsilon\,\mathfrak{h} = 0.$$

Moreover, as in [15, paragraph 2.1(a)] the choice of p ensures that

$$(26) \qquad \epsilon \in \mathfrak{h}^n, \qquad n \geqslant 0.$$

Let $X = \bigcap_n \mathfrak{h}^n R$ and let P be a right ideal of R maximal with respect to not containing ϵ. If $W = R/P$, then (26) shows that WX is non-zero. Let V be the victim of X in W and suppose that U/V is a simple R-submodule of W/V. Let Y be the killer of U/V in R. Since Y is a maximal ideal of R and does not contain $\mathfrak{h}$, it follows that $R = Y + \mathfrak{h} R$ and hence that $R = X + Y$. Therefore ϵ is not in the victim of Y. Hence Y has trivial victim in W. If $XY^\omega \leqslant YX$, then $UXY^\omega = 0$, and we deduce that $UX = 0$. Therefore XY^ω is not contained in YX. If Z is the image of Y under the involution of R determined by $x \to x^{-1}$, $x \in G$, it follows that

$$(27) \qquad Z^{\omega} X \not\leqslant XZ.$$

We now define $M = (XZ + Z^{\omega})/XZ$. (27) shows that $\mathfrak{h}$ does not act nilpotently on M. However, if $\bar{M}$ is a simple image of M, then $\mathfrak{h}$ kills it, for otherwise we should have $\bar{M} = \bar{M}\mathfrak{h}$. It would follow that $\bar{M} = \bar{M}X$, and thence that $\bar{M}Z = 0$. This would give $MZ < M$, which is clearly impossible.

We remark that *if $H = G$, then the module W is monolithic, its lith, by (25), is killed by $\mathfrak{g}$ and yet, by (26), no power of $\mathfrak{g}$ kills W.*

Note added in proof.

A.V. Jategaonkar has recently proved that the monolithic $\mathbf{Z}G$-modules are finite. His proof is in his paper "Integral group rings of polycyclic-by-finite groups", which will be published in this journal. A different proof is given in "Applications of the Artin–Rees Lemma to group rings", which will appear under my name in Symp. Math.

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